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Improved Adomian decomposition method

Tamer A. Abassy *

Department of Scientific Computing, Faculty of Computers and Informatics, Benha University, Benha 13518, Egypt
Department of Mathematics, Faculty of Arts and Science in Wadi Addwasir, King Saud University, Saudi Arabia

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ABSTRACT

In this paper a new treatment for the Adomian decomposition method (ADM) is introduced. The new treatment is called the improved Adomian decomposition method (IADM) which improves the results obtained from the known Adomian decomposition method. The improved Adomian decomposition method is applied for the analytic treatment of nonlinear initial value problems. The improved method accelerates the convergence of the series solution, and provides the exact power series solution. It solves the drawbacks in the standard Adomian decomposition method.

1. Introduction

Nonlinear phenomena, which appear in many areas of scientific fields, can be modeled by partial differential equations. A broad class of analytical solutions methods and numerical solutions methods have been used in to handle these problems, such as the Backlund transformation [1], Hirota's bilinear method [2], the Darboux transformation [3], the symmetry method [4], the inverse scattering transformation [5], the Adomian decomposition method [6,7], recently homotopy perturbation method [8–10], modified variational iteration method [11], and other asymptotic methods have been used to solve nonlinear problems.

The Adomian decomposition method has been proved to be effective and reliable for handling differential equations, linear or nonlinear [6,12–18].

Using the Adomian decomposition method faces some problems with certain types of equation. Wazwaz introduced the modified Adomian decomposition method to solve some of these problems [15].

In this work we introduce a new analytical treatment for nonlinear initial value problems by using the improved Adomian decomposition method. Although the new analytical treatment form introduces a change in the formulation of Adomian polynomials, it provides a qualitative improvement over the standard Adomian method. The improved method can effectively improve the speed of convergence and calculations.

On the other hand, the Boussinesq equation and others can be solved using IADM and ADM. The fourth order Boussinesq equation is a nonlinear initial value problem that reads:

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0, \quad x \in R. \tag{1}$$

The Boussinesq equation (1) describes motions of long waves in shallow water under gravity and in a one dimensional nonlinear lattice [2,19,20]. This particular form (1) is of special interest because [21,22] it admits inverse scattering formalism. This equation also arises in other physical applications such as nonlinear lattice waves, ion sound waves in plasma, and in vibrations in a nonlinear string. Moreover, it has been applied to problems in the percolation of water in porous subsurface strata. For more details about formulation of the Boussinesq equation, see [5,23,24].

* Corresponding address: Department of Scientific Computing, Faculty of Computers and Informatics, Benha University, Benha 13518, Egypt.
E-mail address: tamerabassy@yahoo.com.
URL: http://faculty.ksu.edu.sa/abassy.

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2. The standard Adomian decomposition method and the improved Adomian decomposition method

Consider the following general non-linear initial value problem

$$
\begin{aligned}
&Lu(x,t) + Ru(x,t) + N\left(u(x,t)\right) = 0,\\
&u(x,0) = f_0(x),\\
&\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = f_1(x),\\
&\vdots\\
&\left.\frac{\partial^{s-1} u(x,t)}{\partial t^{s-1}}\right|_{t=0} = \frac{f_{s-1}(x)}{(s-1)!},
\end{aligned}
\tag{2}
$$

where $L = \frac{\partial^s}{\partial t^s}$, $s = 1, 2, 3, \ldots$ is the highest partial derivative with respect to t, R is a linear operator and $Nu(x,t)$ is the nonlinear term. $Ru(x,t)$ and $N\left(u(x,t)\right)$ are free of partial derivatives with respect to t.

Following the usual analysis of standard Adomian [6,25].

The inverse operator L^{-1} is an integral operator which is given by

$$L^{-1}(.) = \int_0^t \ldots \int_0^t (.)\mathrm{d}t \ldots (s\,fold) \ldots \mathrm{d}t. \tag{3}$$

Applying L^{-1} on Eq. (2) and using the constraints leads to

$$u(x,t) = f_0(x) + f_1(x)t + \cdots + f_{s-1}(x)t^{s-1} - L^{-1}\left(N\left(u(x,t)\right) + Ru(x,t)\right). \tag{4}$$

The Adomian decomposition method assumes that the unknown function $u(x,t)$ can be expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{5}$$

and the nonlinear term $N\left(u(x,t)\right)$ can be decomposed by an infinite series of polynomials given by

$$N\left(u(x,t)\right) = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where the components $u_n(x,t)$ will be determined recurrently and A_n are the so-called Adomian polynomials of $u_0, u_1, u_2, \ldots, u_n$ defined by

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3, \ldots. \tag{7}$$

Substituting by Eqs. (5) and (6) into Eq. (4) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f_0(x) + f_1(x)t + \cdots + f_{s-1}(x)t^{s-1} - L^{-1}\left(\sum_{n=0}^{\infty} A_n + R\left(\sum_{n=0}^{\infty} u_n\right)\right). \tag{8}$$

The component of $u_n(x,t)$ follows immediately upon setting

$$u_0(x,t) = f_0(x) + f_1(x)t + \cdots + f_{s-1}(x)t^{s-1} \tag{9}$$

$$u_{n+1}(x,t) = -L^{-1}(A_n + Ru_n), \quad n \geq 0. \tag{10}$$

This standard method is powerful when $s = 1$ but it has some drawbacks like obtaining inaccurate terms when $s = 2, 3, \ldots$. The obtained inaccurate terms consume time in calculation and deteriorate the convergence.

In order to overcome this problem, the Adomian polynomials (7) are redefined in the form

$$
\begin{aligned}
A_n = {} & \frac{1}{ns!}\left[\frac{\mathrm{d}^{ns}}{\mathrm{d}\lambda^{ns}} N\left(\sum_{i=0}^{\infty} \lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(ns+1)!}\left[\frac{\mathrm{d}^{(ns+1)}}{\mathrm{d}\lambda^{(ns+1)}} N\left(\sum_{i=0}^{\infty} \lambda^i f_i t^i\right)\right]_{\lambda=0} + \cdots \\
& + \frac{1}{(ns+s-1)!}\left[\frac{\mathrm{d}^{(ns+s-1)}}{\mathrm{d}\lambda^{(ns+s-1)}} N\left(\sum_{i=0}^{\infty} \lambda^i f_i t^i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3, \ldots,
\end{aligned}
\tag{11}
$$

where f_n is the coefficient of t^n in $u_n(x,t)$ components.

For example when $s = 1$ leads to

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3\ldots, \tag{12}$$

when $s = 2$ leads to

$$A_n = \frac{1}{2n!}\left[\frac{d^{2n}}{d\lambda^{2n}}N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(2n+1)!}\left[\frac{d^{(2n+1)}}{d\lambda^{(2n+1)}}N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3\ldots, \tag{13}$$

when $s = 3$ leads to

$$A_n = \frac{1}{3n!}\left[\frac{d^{3n}}{d\lambda^{3n}}N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(3n+1)!}\left[\frac{d^{(3n+1)}}{d\lambda^{(3n+1)}}N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(3n+2)!}\left[\frac{d^{(3n+2)}}{d\lambda^{(3n+2)}}N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3\ldots, \tag{14}$$

and so on ....

The new Adomian polynomial (11) gives the same result as Adomian polynomials (7) when $s = 1$ but different polynomials when $s = 2, 3, 4, \ldots$.

In the improved Adomian decomposition method (IADM), the Adomian polynomials (11) are used instead of the Adomian polynomials (7). ADM does not give the exact power series for Eq. (2) when $s = 2, 3, 4, \ldots$. IADM gives the exact power series solution for Eq. (2) when $s = 2, 3, 4, \ldots$ and cancels the calculations of all the inaccurate terms which consume time, effort and deteriorate the convergences.

2.1. Illustrative example ($s = 2$)

Consider the "good" Boussinesq equation [15,25–27]

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0, \quad x \in R, \tag{15}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= \frac{-3c^2}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right], \\ u_t(x,0) &= \frac{3c^3\sqrt{1-c^2}}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right] \end{aligned} \tag{16}$$

where c is a constant.

Using IADM

Eq. (15) can be re-written in an operator form as

$$Lu(x,t) + Ru(x,t) + N(u(x,t)) = 0, \tag{17}$$

where the differential operator $L = \frac{\partial^2}{\partial t^2}$, $R = -\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}$, the nonlinear term $N(u(x,t)) = -\left(u^2\right)_{xx}$, and from L we find that $s = 2$.

The inverse operator L^{-1} is an integral operator given by

$$L^{-1}(.) = \int_0^t\int_0^t(.)d\tau d\tau. \tag{18}$$

Applying L^{-1} on Eq. (17) and using the given I.C.'s we find that

$$u(x,t) = f_0(x) + f_1(x)t - L^{-1}(N(u(x,\tau)) + Ru(x,\tau)). \tag{19}$$

Substituting by Eqs. (5) and (6) into Eq. (19) gives

$$\sum_{n=0}^{\infty}u_n(x,t) = f_0(x) + f_1(x)t - L^{-1}\left(\sum_{n=0}^{\infty}A_n(x,\tau) + R\left(\sum_{n=0}^{\infty}u_n(x,\tau)\right)\right), \tag{20}$$

where A_n are the Adomian polynomials which represent the nonlinear term $-\left(u^2\right)_x$ and are defined by (11) where $s = 2$. A_n takes the form

$$A_n = \frac{1}{2n!}\left[\frac{d^{2n}}{d\lambda^{2n}}N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(2n+1)!}\left[\frac{d^{(2n+1)}}{d\lambda^{(2n+1)}}N\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3\ldots. \tag{21}$$

Using

$$u_0(x,t)=f_0(x)+f_1(x)t, \tag{22}$$

and the iteration formula

$$u_{n+1}(x,t)=-L^{-1}(A_n+Ru_n),\quad n\geq 0 \tag{23}$$

leads to the following results

$$\begin{aligned}
u_0 &= \frac{-3c^2}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right]+\frac{3c^3\sqrt{1-c^2}}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right]t\\
&= f_0(x)+f_1(x)t
\end{aligned}$$

$$f_0(x)=\frac{-3c^2}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right],$$

$$f_1(x)=\frac{3c^3\sqrt{1-c^2}}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right].$$

$$A_0=2(f_{0_x})^2+2f_0f_{0_{xx}}+\left(4f_{0_x}f_{1_x}+2f_1f_{0_{xx}}+2f_0f_{1_{xx}}\right)t,$$

$$\begin{aligned}
u_1(x,t) &= -L^{-1}(A_0+Ru_0)\\
&= \frac{3}{8}c^4(-1+c^2)\,(-2+\cosh[cx])\operatorname{sech}^4\left[\frac{cx}{2}\right]t^2\\
&\quad+\frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\operatorname{sech}^5\left[\frac{cx}{2}\right]\left(-11\sinh\left[\frac{cx}{2}\right]+\sinh\left[\frac{3cx}{2}\right]\right)\right)t^3\\
&= f_2(x)t^2+f_3(x)t^3,
\end{aligned}$$

$$f_2(x)=\frac{3}{8}c^4(-1+c^2)\,(-2+\cosh[cx])\operatorname{sech}^4\left[\frac{cx}{2}\right]t^2,$$

$$f_3(x)=\frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\operatorname{sech}^5\left[\frac{cx}{2}\right]\left(-11\sinh\left[\frac{cx}{2}\right]+\sinh\left[\frac{3cx}{2}\right]\right)\right)t^3,$$

$$A_1=2(f_{1_x}^2+2f_{0_x}f_{2_x}+f_{0_{xx}}f_2+f_1f_{1_{xx}}+f_{2_{xx}}f_0)t^2+2(2f_{1_x}f_{2_x}+2f_{0_x}f_{3_x}+f_{0_{xx}}f_3+f_{1_{xx}}f_2+f_{2_{xx}}f_1+f_{3_{xx}}f_0)t^3,$$

$$\begin{aligned}
u_2(x,t) &= -L^{-1}(A_1+Ru_1)\\
&= -\frac{1}{128}c^6(-1+c^2)^2\,(33-26\cosh[cx]+\cosh[2cx])\operatorname{sech}^6\left[\frac{cx}{2}\right]t^4\\
&\quad+\frac{c^7(1-c^2)^{\frac{5}{2}}}{1280}\left(\operatorname{sech}^7\left[\frac{cx}{2}\right]\left(302\sinh\left[\frac{cx}{2}\right]-57\sinh\left[\frac{3cx}{2}\right]+\sinh\left[\frac{5cx}{2}\right]\right)\right)t^5\\
&= f_4(x)t^4+f_5(x)t^5,\\
&\ \ \vdots
\end{aligned}$$

Following the same procedures we obtain

$$\begin{aligned}
u_3(x,t) &= -L^{-1}(A_2+Ru_2)\\
&= \frac{c^8(-1+c^2)^3}{15360}(-1208+1191\cosh[cx]-120\cosh[2cx]+\cosh[3cx])\operatorname{sech}^8\left[\frac{cx}{2}\right]t^6-\frac{c^9(1-c^2)^{\frac{7}{2}}}{215040}\\
&\quad\times\operatorname{sech}^9\left[\frac{cx}{2}\right]\left(-15619\sinh\left[\frac{cx}{2}\right]+4293\sinh\left[\frac{3cx}{2}\right]-247\sinh\left[\frac{5cx}{2}\right]+\sinh\left[\frac{7cx}{2}\right]\right)t^7,
\end{aligned}$$

And so on….

Considering these components, the solution can be approximated as:

$$u(x,t)\simeq\phi_n(x,t)=\sum_{m=0}^{n}u_m(x,t). \tag{24}$$

$$\begin{aligned}
\phi_1 &= \frac{-3c^2}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right]+\frac{3c^3\sqrt{1-c^2}}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right]t+\frac{3}{8}c^4(-1+c^2)\,(-2+\cosh[cx])\operatorname{sech}^4\left[\frac{cx}{2}\right]t^2\\
&\quad+\frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\operatorname{sech}^5\left[\frac{cx}{2}\right]\left(-11\sinh\left[\frac{cx}{2}\right]+\sinh\left[\frac{3cx}{2}\right]\right)\right)t^3,
\end{aligned}$$

$$
\begin{aligned}
\phi_2 &= \frac{-3c^2}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right] + \frac{3c^3\sqrt{1-c^2}}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right]t + \frac{3}{8}c^4(-1+c^2)\,(-2+\cosh[cx])\operatorname{sech}^4\left[\frac{cx}{2}\right]t^2 \\
&\quad + \frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\operatorname{sech}^5\left[\frac{cx}{2}\right]\left(-11\sinh\left[\frac{cx}{2}\right] + \sinh\left[\frac{3cx}{2}\right]\right)\right)t^3 \\
&\quad - \frac{1}{128}c^6(-1+c^2)^2(33 - 26\cosh[cx] + \cosh[2cx])\operatorname{sech}^6\left[\frac{cx}{2}\right]t^4 \\
&\quad + \frac{c^7(1-c^2)^{5/2}}{1280}\left(\operatorname{sech}^7\left[\frac{cx}{2}\right]\left(302\sinh\left[\frac{cx}{2}\right] - 57\sinh\left[\frac{3cx}{2}\right] + \sinh\left[\frac{5cx}{2}\right]\right)\right)t^5, \qquad (25) \\
\phi_3 &= \frac{-3c^2}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right] + \frac{3c^3\sqrt{1-c^2}}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right]\ t + \frac{3}{8}c^4(-1+c^2)\,(-2+\cosh[cx])\operatorname{sech}^4\left[\frac{cx}{2}\right]t^2 \\
&\quad + \frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\operatorname{sech}^5\left[\frac{cx}{2}\right]\left(-11\sinh\left[\frac{cx}{2}\right] + \sinh\left[\frac{3cx}{2}\right]\right)\right)t^3 \\
&\quad - \frac{1}{128}c^6(-1+c^2)^2\,(33 - 26\cosh[cx] + \cosh[2cx])\operatorname{sech}^6\left[\frac{cx}{2}\right]t^4 \\
&\quad + \frac{c^7(1-c^2)^{\frac{5}{2}}}{1280}\left(\operatorname{sech}^7\left[\frac{cx}{2}\right]\left(302\sinh\left[\frac{cx}{2}\right] - 57\sinh\left[\frac{3cx}{2}\right] + \sinh\left[\frac{5cx}{2}\right]\right)\right)t^5 \\
&\quad + \frac{c^8(-1+c^2)^3}{15360}(-1208 + 1191\cosh[cx] - 120\cosh[2cx] + \cosh[3cx])\operatorname{sech}^8\left[\frac{cx}{2}\right]t^6 \\
&\quad - \frac{(c^9(1-c^2))^{\frac{7}{2}}}{215040}\operatorname{sech}^9\left[\frac{cx}{2}\right]\left(-15619\sinh\left[\frac{cx}{2}\right] + 4293\sinh\left[\frac{3cx}{2}\right] - 247\sinh\left[\frac{5cx}{2}\right] + \sinh\left[\frac{7cx}{2}\right]\right)t^7, \\
&\ \ \vdots
\end{aligned}
$$

ϕ_n contains the exact power series expansion of the closed form solution

$$
u(x,t) = \frac{-3c^2}{2}\operatorname{sech}^2\left[\frac{c}{2}(x + \sqrt{1-c^2}\ t)\right]. \qquad (26)
$$

Using ADM

The components $u_n(x,t)$ follow immediately upon setting

$$
\begin{aligned}
u_0(x,t) &= f_0(x) + f_1(x)t \\
&= \frac{-3c^2}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right] + \frac{3c^3\sqrt{1-c^2}}{2}\operatorname{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right]t, \qquad (27)
\end{aligned}
$$

and using the iterative equation (10) where A_n are the Adomian polynomials of $u_0, u_1, \ldots, u_n$, represent the nonlinear term $-\left(u^2\right)_{xx}$ and is defined by (7) see [25].

The following components are obtained

$$
\begin{aligned}
u_1(x,t) &= \frac{3}{8}c^4(-1+c^2)\,(-2+\cosh[cx])\operatorname{sech}^4\left[\frac{cx}{2}\right]t^2 + \frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\operatorname{sech}^5\left[\frac{cx}{2}\right]\left(-11\sinh\left[\frac{cx}{2}\right]\right.\right. \\
&\quad \left.\left. +\ \sinh\left[\frac{3cx}{2}\right]\right)\right)t^3 - \frac{3}{32}c^8(-1+c^2)\,(10 - 10\cosh[cx] + \cosh[2cx])\operatorname{sech}^6\left[\frac{cx}{2}\right]t^4, \\
u_2(x,t) &= -\frac{1}{512}c^6(-1+c^2)(-40 - 440c^2 + 15(-1+33c^2)\cosh[cx] + 24(-1+3c^2)\cosh[2cx] \\
&\quad + (-1+c^2)\cosh[3cx])\operatorname{sech}^8\left[\frac{cx}{2}\right]t^4 + \frac{c^7(1-c^2)^{\frac{5}{2}}}{1280}\left(\operatorname{sech}^7\left[\frac{cx}{2}\right]\left(302\sinh\left[\frac{cx}{2}\right] - 57\sinh\left[\frac{3cx}{2}\right]\right.\right. \\
&\quad \left.\left. +\ \sinh\left[\frac{5cx}{2}\right]\right)\right)t^5 + \frac{1}{2560}c^{10}(-1+c^2)(600 + 26670c^2 - (59 + 33775c^2)\cosh[cx] \\
&\quad + 4(-133 + 2050c^2)\cosh[2cx] + (123 - 645c^2)\cosh[3cx] - (4 - 10c^2)\cosh[4cx])\operatorname{sech}^{12}\left[\frac{cx}{2}\right]t^6 \\
&\quad + \frac{3c^{13}(1-c^2)^{\frac{3}{2}}}{3584}\left(-3749\sinh\left[\frac{cx}{2}\right] + 1551\sinh\left[\frac{3cx}{2}\right] - 235\sinh\left[\frac{5cx}{2}\right] + 9\sinh\left[\frac{7cx}{2}\right]\right)\operatorname{sech}^{13}\left[\frac{cx}{2}\right]t^7,
\end{aligned}
$$

$$u_3(x,t) = \frac{1}{245760}c^8(-1+c^2)(-2604-52392c^2-2562924c^4+6(-301+1546c^2+540099c^4)\cosh[cx] \\ +96(-17-498c^2+8183c^4)\cosh[2cx]+(717-13242c^2+62637c^4)\cosh[3cx] \\ -(116-616c^2+1076c^4)\cosh[4cx]+(1-2c^2+c^4)\cosh[5cx])\,\text{sech}^{12}\left[\frac{cx}{2}\right]t^6 \\ +\frac{c^9(1-c^2)^{3/2}}{1720320}(-18774+37548c^2+3471786c^4-6(3157-6314c^2+639157c^4)\cosh[cx] \\ +24(131-262c^2+27251c^4)\cosh[2cx]+(3069-6138c^2+22851c^4)\cosh[3cx] \\ -(242-484c^2+2426c^4)\cosh[4cx]+(1-2c^2+c^4)\cosh[5cx])\,\text{sech}^{12}\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right]t^7 \\ +\frac{1}{1146880}c^{12}(-1+c^2)(-23856-1341312c^2-11811806c^4+6(-1249+97406c^2 \\ +27630223c^4)\cosh[cx]+(24612+1224108c^2+56201988c^4)\cosh[2cx]+(4157-614002c^2 \\ +8634497c^4)\cosh[3cx]+(-3760+84416c^2-526384c^4)\cosh[4cx]+(329-3202c^2 \\ +9701c^4)\cosh[5cx]+(-4+20c^2-28c^4)\cosh[6cx])\,\text{sech}^{16}\left[\frac{cx}{2}\right]t^8+\ldots,$$

$$\vdots$$

Considering these components, the solution can be approximated as:

$$u(x,t) \simeq \phi_n(x,t) = \sum_{m=0}^{n} u_m(x,t). \tag{28}$$

$$\phi_1 = \frac{-3c^2}{2}\text{sech}^2\left[\frac{cx}{2}\right]+\frac{3c^3\sqrt{1-c^2}}{2}\text{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right]\;t+\frac{3}{8}c^4(-1+c^2)(-2+\cosh[cx])\,\text{sech}^4\left[\frac{cx}{2}\right]t^2 \\ +\frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\,\text{sech}^5\left[\frac{cx}{2}\right]\left(-11\sinh\left[\frac{cx}{2}\right]+\sinh\left[\frac{3cx}{2}\right]\right)\right)t^3 \\ \underline{-\frac{3}{32}c^8(-1+c^2)\,(10-10\cosh[cx]+\cosh[2cx])\,\text{sech}^6\left[\frac{cx}{2}\right]t^4},$$

$$\phi_2 = \frac{-3c^2}{2}\text{sech}^2\left[\frac{cx}{2}\right]+\frac{3c^3\sqrt{1-c^2}}{2}\text{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right]\;t+\frac{3}{8}c^4(-1+c^2)\,(-2+\cosh[cx])\,\text{sech}^4\left[\frac{cx}{2}\right]t^2 \\ +\frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\,\text{sech}^5\left[\frac{cx}{2}\right]\left(-11\sinh\left[\frac{cx}{2}\right]+\sinh\left[\frac{3cx}{2}\right]\right)\right)t^3-\frac{1}{128}c^6(-1+c^2)^2\,(33-26\cosh[cx] \\ +\cosh[2cx])\,\text{sech}^6\left[\frac{cx}{2}\right]t^4+\frac{c^7(1-c^2)^{5/2}}{1280}\left(\text{sech}^7\left[\frac{cx}{2}\right]\left(302\sinh\left[\frac{cx}{2}\right]-57\sinh\left[\frac{3cx}{2}\right]\right.\right. \\ \left.\left.+\sinh\left[\frac{5cx}{2}\right]\right)\right)t^5+\underline{\frac{1}{2560}c^{10}(-1+c^2)(600+26670c^2-(59+33775c^2)\cosh[cx]+4(-133} \\ \underline{+2050c^2)\cosh[2cx]+(123-645c^2)\cosh[3cx]-(4-10c^2)\cosh[4cx])\,\text{sech}^{12}\left[\frac{cx}{2}\right]t^6}+\ldots, \tag{29}$$

$$\phi_3 = \frac{-3c^2}{2}\text{sech}^2\left[\frac{cx}{2}\right]+\frac{3c^3\sqrt{1-c^2}}{2}\text{sech}^2\left[\frac{cx}{2}\right]\tanh\left[\frac{cx}{2}\right]\;t+\frac{3}{8}c^4(-1+c^2)\,(-2+\cosh[cx])\,\text{sech}^4\left[\frac{cx}{2}\right]t^2 \\ +\frac{1}{16}\left(c^5(1-c^2)^{\frac{3}{2}}\,\text{sech}^5\left[\frac{cx}{2}\right]\left(-11\sinh\left[\frac{cx}{2}\right]+\sinh\left[\frac{3cx}{2}\right]\right)\right)t^3 \\ -\frac{1}{128}c^6(-1+c^2)^2\,(33-26\cosh[cx]+\cosh[2cx])\,\text{sech}^6\left[\frac{cx}{2}\right]t^4+\frac{c^7(1-c^2)^{\frac{5}{2}}}{1280}\left(\text{sech}^7\left[\frac{cx}{2}\right]\right. \\ \times\left.\left(302\sinh\left[\frac{cx}{2}\right]-57\sinh\left[\frac{3cx}{2}\right]+\sinh\left[\frac{5cx}{2}\right]\right)\right)t^5+\frac{c^8(-1+c^2)^3}{15360}(-1208+1191\cosh[cx] \\ -120\cosh[2cx]+\cosh[3cx])\,\text{sech}^8\left[\frac{cx}{2}\right]t^6-\frac{c^9(1-c^2)^{\frac{7}{2}}}{215040}\left(\text{sech}^9\left[\frac{cx}{2}\right]\left(-15619\sinh\left[\frac{cx}{2}\right]\right.\right. \\ \left.\left.+4293\sinh\left[\frac{3cx}{2}\right]-247\sinh\left[\frac{5cx}{2}\right]+\sinh\left[\frac{7cx}{2}\right]\right)\right)t^7+\underline{\frac{1}{1146880}c^{12}(-1+c^2)(-23856} \\ \underline{-1341312c^2-11811806c^4+6(-1249+97406c^2+27630223c^4)\cosh[cx]+(24612+1224108c^2} \\ \underline{+56201988c^4)\cosh[2cx]+(4157-614002c^2+8634497c^4)\cosh[3cx]+(-3760+84416c^2-526384c^4)}$$

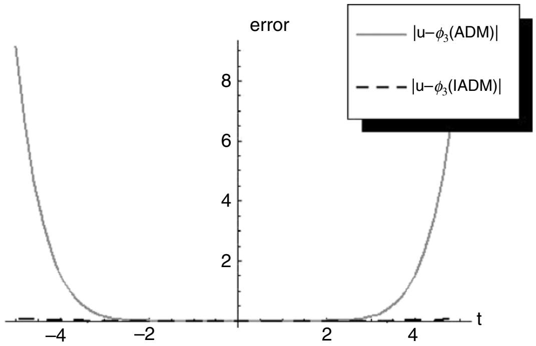

Fig. 1. The absolute error between the closed form solution (26) and $\phi_{3_{(ADM)}} = \phi_n^{ST} + \phi_3^{NS}$, and the absolute error between the closed form solution (26) and $\phi_{3_{IADM}} = \phi_n^{ST}$ at $x = 0$ and $c = 0.5$.

$$\underline{\times \cosh[4cx] + (329 - 3202c^2 + 9701c^4)\cosh[5cx] + (-4 + 20c^2 - 28c^4)\cosh[6cx])\,\mathrm{sech}^{16}\left[\frac{cx}{2}\right] t^8 + \cdots,}$$

$$\vdots$$

In $\phi_{2_{(ADM)}}$ and $\phi_{3_{(ADM)}}$ some terms are not written because it is massive to write them.

Results analysis

From analyzing the previous results, it can be noticed that the results of ADM without improvement take the following form:

$$\phi_{n_{(ADM)}} = f_n^0 + f_n^1 t + f_n^2 t^2 + \cdots + f_n^{2n+1} t^{2n+1} \underline{+ f_n^{2n+2} t^{2n+2} + f_n^{2n+3} t^{2n+3} + \cdots}, \tag{30}$$

where f_n^m is the coefficient of t^m. f_n^m is settled, accurate, and takes the same value for each $\phi_{n_{(ADM)}}$ as $(2n + 1) \geq m$. f_n^m is not settled, not accurate, and does not take the same value for each $\phi_{n_{(ADM)}}$ as $(2n + 1) < m$ i.e. $\phi_{n_{(ADM)}}$ can be rewritten in the form

$$\phi_{n_{(ADM)}}(x, t) = \phi_n^{ST}(x, t) + \phi_n^{NS}(x, t), \tag{31}$$

where ϕ_n^{ST} contains the settled terms (accurate terms) in Eq. (30) (not underlined terms) and ϕ_n^{NS} contains the non-settled terms (not accurate terms) in Eq. (30) (underlined terms). In Fig. 1, we show that the addition of the term $\phi_n^{NS}(x, t)$ deteriorates the convergence to the closed form solution (26) since the coefficients of t^s in $\phi_n^{NS}(x, t)$are not the exact coefficients of t^s. ϕ_n^{ST}, which shows better results, is the same as the result obtained from IADM.

2.2. Illustrative example ($s = 3$)

Consider the nonlinear initial value problem

$$\begin{aligned} &\frac{d^3u(x,t)}{dt^3} + \frac{du(x,t)}{dx} - 2x\,(u(x,t))^2 + 6\,(u(x,t))^4 = 0 \\ &u(x,0) = \frac{-1}{x^2}, \qquad u_t(x,0) = \frac{1}{x^4}, \qquad u_{tt}(x,0) = \frac{-1}{x^6}. \end{aligned} \tag{32}$$

Solving Eq. (32) using IADM, Eq. (32) is written in the form:

$$Lu(x, t) + Ru(x, t) + N1(u(x, t)) + N2(u(x, t)) = 0, \tag{33}$$

where $Lu(x,t) = \frac{d^3u(x,t)}{dt^3}$, $Ru(x,t) = \frac{du(x,t)}{dx}$, $N1(u(x,t)) = -2x\,(u(x,t))^2$, $N2(u(x,t)) = 6\,(u(x,t))^4$, and from L we find that $s = 3$.

The inverse operator L^{-1} is an integral operator given by

$$L^{-1}(.) = \int_0^t \int_0^t \int_0^t (.)\,d\tau\, d\tau\, d\tau. \tag{34}$$

Applying L^{-1} on Eq. (33) and using the given I.C.'s we find that

$$u(x, t) = f_0(x) + f_1(x)t + f_2(x)t^2 - L^{-1}(N1(u(x, \tau)) + N2(u(x, \tau)) + Ru(x, \tau)). \tag{35}$$

Substituting by Eq. (5) into Eq. (35) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f_0(x) + f_1(x)t + f_2(x)t^2 - L^{-1}\left(\sum_{n=0}^{\infty} A_n(x,\tau) + \sum_{n=0}^{\infty} B_n(x,\tau) + R\left(\sum_{n=0}^{\infty} u_n(x,\tau)\right)\right), \tag{36}$$

where A_n and B_n are the Adomian polynomials which represent the nonlinear terms $N1(u(t,x)) = -2x\,(u(t,x))^2$ and $N2(u(t,x)) = 6\,(u(t,x))^4$ respectively and are defined by

$$\begin{aligned} A_n &= \frac{1}{3n!}\left[\frac{d^{3n}}{d\lambda^{3n}} N1\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(3n+1)!}\left[\frac{d^{(3n+1)}}{d\lambda^{(3n+1)}} N1\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} \\ &\quad + \frac{1}{(3n+2)!}\left[\frac{d^{(3n+2)}}{d\lambda^{(3n+2)}} N1\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3\ldots, \end{aligned} \tag{37}$$

$$\begin{aligned} B_n &= \frac{1}{3n!}\left[\frac{d^{3n}}{d\lambda^{3n}} N2\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(3n+1)!}\left[\frac{d^{(3n+1)}}{d\lambda^{(3n+1)}} N2\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} \\ &\quad + \frac{1}{(3n+2)!}\left[\frac{d^{(3n+2)}}{d\lambda^{(3n+2)}} N2\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0}, \; n = 0, 1, 2, 3\ldots, \end{aligned} \tag{38}$$

and using

$$u_0(x,t) = f_0(x) + f_1(x)t + f_2(x)t^2 \tag{39}$$

$$u_{n+1}(x,t) = L^{-1}(A_n + B_n + Ru_n), \quad n \geq 0 \tag{40}$$

the following results are obtained

$$\begin{aligned} u_0 &= -\frac{1}{x^2} - \frac{1}{x^4}t - \frac{1}{x^6}t^2 \\ &= f_0(x) + f_1(x)t + f_2(x)t^2, \end{aligned}$$

$$f_0(x) = -\frac{1}{x^2}, \qquad f_1(x) = -\frac{1}{x^4}, \qquad f_2(x) = -\frac{1}{x^6}.$$

$$A_0 = -2x\left(f_0^2 + 2f_0f_1t + \left(f_1^2 + 2f_0f_2\right)t^2\right),$$

$$B_0 = 6\left(f_0^4 + 4f_0^3f_1t + 2f_0^2(3f_1^2 + 2f_0f_2)t^2\right)$$

$$\begin{aligned} u_1(x,t) &= -L^{-1}(A_0 + B_0 + Ru_0) \\ &= -\frac{1}{x^8}t^3 - \frac{1}{x^{10}}t^4 - \frac{1}{x^{12}}t^5 \\ &= f_3(x)t^3 + f_4(x)t^4 + f_5(x)t^5, \end{aligned}$$

$$f_3(x) = -\frac{1}{x^8}, \qquad f_4(x) = -\frac{1}{x^{10}}, \qquad f_5(x) = -\frac{1}{x^{12}}.$$

$$A_1 = -2x(2(f_1f_2 + f_0f_3)t^3 + (f_2^2 + 2f_1f_3 + 2f_0f_4)t^4 + 2(f_2f_3 + f_1f_4 + f_0f_5)t^5)$$

$$\begin{aligned} B_1 &= 4(6f_0f_1^3 + 18f_0^2f_1f_2 + 6f_0^3f_3)t^3 + (6f_1^4 + 72f_0f_1^2f_2 + 36f_0^2f_2^2 + 72f_0^2f_1f_3 + 24f_0^3f_4)t^4 \\ &\quad + 24(f_1^3f_2 + 3f_0f_1f_2^2 + 3f_0f_1^2f_3 + 3f_0^2f_2f_3 + 3f_0^2f_1f_4 + f_0^3f_5)t^5 \end{aligned}$$

$$\begin{aligned} u_2(x,t) &= -L^{-1}(A_1 + B_1 + Ru_1) \\ &= -\frac{1}{x^{14}}t^6 - \frac{1}{x^{16}}t^7 - \frac{1}{x^{18}}t^8 \\ &= f_6(x)t^6 + f_7(x)t^7 + f_8(x)t^8, \end{aligned}$$

following the same procedures we obtain

$$\begin{aligned} u_3(x,t) &= L^{-1}(A_2 + B_2 + Ru_2) \\ &= -\frac{1}{x^{20}}t^9 - \frac{1}{x^{22}}t^{10} - \frac{1}{x^{24}}t^{11}, \\ &\;\;\vdots \end{aligned}$$

And so on. . . .

Considering these components, the solution can be approximated as:

$$u(x,t) \simeq \phi_n(x,t) = \sum_{m=0}^{n} u_m(x,t). \tag{41}$$

$$\begin{aligned}
\phi_1 &= -\frac{1}{x^2} - \frac{1}{x^4}t - \frac{1}{x^6}t^2 - \frac{1}{x^8}t^3 - \frac{1}{x^{10}}t^4 - \frac{1}{x^{12}}t^5,\\
\phi_2 &= -\frac{1}{x^2} - \frac{1}{x^4}t - \frac{1}{x^6}t^2 - \frac{1}{x^8}t^3 - \frac{1}{x^{10}}t^4 - \frac{1}{x^{12}}t^5 - \frac{1}{x^{14}}t^6 - \frac{1}{x^{16}}t^7 - \frac{1}{x^{18}}t^8,\\
\phi_3 &= -\frac{1}{x^2} - \frac{1}{x^4}t - \frac{1}{x^6}t^2 - \frac{1}{x^8}t^3 - \frac{1}{x^{10}}t^4 - \frac{1}{x^{12}}t^5 - \frac{1}{x^{14}}t^6 - \frac{1}{x^{16}}t^7 - \frac{1}{x^{18}}t^8 - \frac{1}{x^{20}}t^9 - \frac{1}{x^{22}}t^{10} - \frac{1}{x^{24}}t^{11},\\
&\vdots
\end{aligned} \tag{42}$$

ϕ_n contains the exact power series expansion of the closed form solution

$$u(t,x) = \frac{1}{t-x^2}. \tag{43}$$

Using ADM

The components $u_n(x,t)$ follow immediately upon setting

$$\begin{aligned}
u_0(x,t) &= f_0(x) + f_1(x)t + f_2(x)t^2\\
&= -\frac{1}{x^2} - \frac{1}{x^4}t - \frac{1}{x^6}t^2,
\end{aligned} \tag{44}$$

and using the iterative equation

$$u_{n+1}(x,t) = -L^{-1}(A_n + B_n + Ru_n), \quad n \geq 0 \tag{45}$$

where A_n and B_n are the Adomian polynomials of $u_0, u_1, \ldots, u_n$, represent the nonlinear term $N1(u(t,x)) = -2x\,(u(t,x))^2$ and $N2(u(t,x)) = 6\,(u(t,x))^4$ respectively and are defined by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N1\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3\ldots, \tag{46}$$

$$B_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N2\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3\ldots, \tag{47}$$

the following components are obtained

$$\begin{aligned}
u_1(x,t) &= -\frac{1}{x^8}t^3 - \frac{1}{x^{10}}t^4 - \frac{1}{x^{12}}t^5 + \frac{-24+x^5}{30x^{14}}t^6 + \frac{-57+x^5}{105x^{16}}t^7 - \frac{2}{7x^{18}}t^8 - \frac{5}{42x^{20}}t^9 - \frac{1}{30x^{22}}t^{10} - \frac{1}{165x^{24}}t^{11},\\
u_2(x,t) &= \frac{-6-x^5}{30x^{14}}t^6 - \frac{48+x^5}{105x^{16}}t^7 - \frac{5}{7x^{18}}t^8 + \frac{(-12096+24x^5+5x^{10})}{15120x^{20}}t^9\\
&\quad - \frac{(55296-348x^5+7x^{10})}{75600x^{22}}t^{10} - \frac{(3180-40x^5+x^{10})}{5775x^{24}}t^{11} + \cdots,\\
u_3(x,t) &= -\frac{(1224+24x^5+5x^{10})}{15120x^{20}}t^9 - \frac{(17784+348x^5-7x^{10})}{75600x^{22}}t^{10} - \frac{(2560+40x^5-x^{10})}{5775x^{24}}t^{11}\\
&\quad - \frac{(2025216+3640064x^5-38823x^{10})}{3326400x^{26}}t^{12} - \frac{(7430832-12432x^5+80223x^{10})}{10810800x^{28}}t^{13} + \cdots,\\
&\vdots
\end{aligned} \tag{48}$$

Considering these components, the solution can be approximated as:

$$u(x,t) \simeq \phi_n(x,t) = \sum_{m=0}^{n} u_m(x,t). \tag{49}$$

$$\begin{aligned}
\phi_1 &= -\frac{1}{x^2} - \frac{1}{x^4}t - \frac{1}{x^6}t^2 - \frac{1}{x^8}t^3 - \frac{1}{x^{10}}t^4 - \frac{1}{x^{12}}t^5 + \underline{\frac{-24+x^5}{30x^{14}}t^6 + \frac{-57+x^5}{105x^{16}}t^7}\\
&\quad \underline{- \frac{2}{7x^{18}}t^8 - \frac{5}{42x^{20}}t^9 - \frac{1}{30x^{22}}t^{10} - \frac{1}{165x^{24}}t^{11}},
\end{aligned}$$

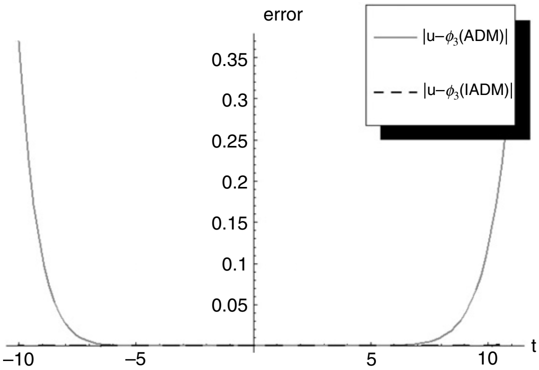

Fig. 2. The absolute error between the closed form solution (43) and $\phi_{3_{(ADM)}} = \phi_n^{ST} + \phi_3^{NS}$, and the absolute error between the closed form solution (43) and $\phi_{3_{IADM}}^{=} \phi_n^{ST}$ at $x = 4$.

$$\begin{aligned}
\phi_2 &= -\frac{1}{x^2} - \frac{1}{x^4}t - \frac{1}{x^6}t^2 - \frac{1}{x^8}t^3 - \frac{1}{x^{10}}t^4 - \frac{1}{x^{12}}t^5 - \frac{1}{x^{14}}t^6 - \frac{1}{x^{16}}t^7 - \frac{1}{x^{18}}t^8 - \underline{\left(\frac{+13896 - 24x^5 - 5x^{10}}{15120x^{20}}\right)t^9} \\
&\quad \underline{- \left(\frac{1}{30x^{22}} + \frac{(55296 - 348x^5 + 7x^{10})}{75600x^{22}}\right)t^{10} - \left(\frac{1}{165x^{24}} + \frac{(3180 - 40x^5 + x^{10})}{5775x^{24}}\right)t^{11} + \cdots}, \\
\phi_3 &= -\frac{1}{x^2} - \frac{1}{x^4}t - \frac{1}{x^6}t^2 - \frac{1}{x^8}t^3 - \frac{1}{x^{10}}t^4 - \frac{1}{x^{12}}t^5 - \frac{1}{x^{14}}t^6 - \frac{1}{x^{16}}t^7 - \frac{1}{x^{18}}t^8 - \frac{1}{x^{20}}t^9 - \frac{1}{x^{22}}t^{10} - \frac{1}{x^{24}}t^{11} \\
&\quad \underline{+ \left(\frac{-3224736 + 7392x^5 - 272x^{10} + 5x^{15}}{3326400x^{26}}\right)t^{12} - \left(\frac{9654192 - 59172x^5 + 607x^{10} + 13x^{15}}{10810800x^{28}}\right)t^{13} + \cdots}, \\
&\ \ \vdots
\end{aligned} \tag{50}$$

It can be seen in Fig. 2 how much the result improved after using IADM.

2.3. Illustrative example ($s = 4$)

Consider the nonlinear initial value problem

$$\begin{aligned}
&\frac{d^4u(t)}{dt^4} - 24\,(u(t))^5 - 40\,(u(t))^3 - 16u(t) = 0 \\
&u(0) = 0, \qquad u_t(0) = 1, \qquad u_{tt}(0) = 0, \qquad u_{ttt}(0) = \frac{1}{3}.
\end{aligned} \tag{51}$$

Solving Eq. (51) using IADM, Eq. (51) is written in the form:

$$Lu(x,t) + Ru(x,t) + N1(u(x,t)) + N2(u(x,t)) = 0, \tag{52}$$

where $Lu(x,t) = \frac{d^4u(x,t)}{dt^4}$, $Ru(t) = -16u(t)$, $N1(u(t,x)) = -40\,(u(t))^3$, $N2(u(t,x)) = -24\,(u(t))^5$, and from L we find that $s = 4$.

The inverse operator L^{-1} is an integral operator given by

$$L^{-1}(.) = \int_0^t \int_0^t \int_0^t \int_0^t (.)d\tau d\tau d\tau d\tau. \tag{53}$$

Applying L^{-1} on Eq. (52) and using the given I.C.'s we find that

$$u(x,t) = f_0(x) + f_1(x)t + f_2(x)t^2 - L^{-1}(N1(u(x,\tau)) + N2(u(x,\tau)) + Ru(x,\tau)). \tag{54}$$

Substituting by Eqs. (5) into Eq. (54) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f_0(x) + f_1(x)t + f_2(x)t^2 + f_3(x)t^3 - L^{-1}\left(\sum_{n=0}^{\infty} A_n(x,\tau) + \sum_{n=0}^{\infty} B_n(x,\tau) + R\left(\sum_{n=0}^{\infty} u_n(x,\tau)\right)\right), \tag{55}$$

where A_n and B_n are the Adomian polynomials which represent the nonlinear terms $N1(u(t,x)) = -40\,(u(t))^3$, and $N2(u(t,x)) = -24\,(u(t))^5$ respectively and defined by

$$\begin{aligned} A_n &= \frac{1}{4n!}\left[\frac{\mathrm{d}^{4n}}{\mathrm{d}\lambda^{4n}}N1\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(4n+1)!}\left[\frac{\mathrm{d}^{(4n+1)}}{\mathrm{d}\lambda^{(4n+1)}}N1\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} \\ &\quad + \frac{1}{(4n+2)!}\left[\frac{\mathrm{d}^{(4n+2)}}{\mathrm{d}\lambda^{(4n+2)}}N1\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(4n+3)!}\left[\frac{\mathrm{d}^{(4n+3)}}{\mathrm{d}\lambda^{(4n+3)}}N1\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0}, \\ &\quad n = 0, 1, 2, 3\ldots, \end{aligned} \tag{56}$$

$$\begin{aligned} B_n &= \frac{1}{4n!}\left[\frac{\mathrm{d}^{4n}}{\mathrm{d}\lambda^{4n}}N2\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(4n+1)!}\left[\frac{\mathrm{d}^{(4n+1)}}{\mathrm{d}\lambda^{(4n+1)}}N2\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} \\ &\quad + \frac{1}{(4n+2)!}\left[\frac{\mathrm{d}^{(4n+2)}}{\mathrm{d}\lambda^{(4n+2)}}N2\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} + \frac{1}{(4n+3)!}\left[\frac{\mathrm{d}^{(4n+3)}}{\mathrm{d}\lambda^{(4n+3)}}N2\left(\sum_{i=0}^{\infty}\lambda^i f_i t^i\right)\right]_{\lambda=0} \\ &\quad n = 0, 1, 2, 3\ldots, \end{aligned} \tag{57}$$

and using

$$u_0(x,t) = f_0(x) + f_1(x)t + f_2(x)t^2 + f_3(x)t^3, \tag{58}$$

$$u_{n+1}(x,t) = -L^{-1}(A_n + B_n + Ru_n), \quad n \geq 0 \tag{59}$$

the following results are obtained

$$\begin{aligned} u_0 &= t + \frac{1}{3}t^3 \\ &= f_0(x) + f_1(x)t^1 + f_2(x)t^2 + f_3(x)t^3, \end{aligned}$$

$$f_0(x) = 0, \qquad f_1(x) = 1, \qquad f_2(x) = 0, \qquad f_3(x) = \frac{1}{3}.$$

$$A_0 = -40(f_0^3 + 3f_0^2 f_1 t + 3f_0(f_1^2 + f_0 f_2)t^2 + (f_1^3 + 6f_0 f_1 f_2 + 3f_0^2 f_3)t^3)$$

$$B_0 = -24(f_0^5 + 5f_0^4 f_1 t + (10f_0^3 f_1^2 + 5f_0^4 f_2)t^2 + 5f_0^2(2f_1^3 + 4f_0 f_1 f_2 + f_0^2 f_3)t^3)$$

$$\begin{aligned} u_1(x,t) &= -L^{-1}(A_0 + B_0 + Ru_0) \\ &= \frac{2}{15}t^5 + \frac{17}{315}t^7 \\ &= f_4(x)t^4 + f_5(x)t^5 + f_6(x)t^6 + f_7(x)t^7, \end{aligned}$$

$$f_4(x) = 0, \qquad f_5(x) = \frac{2}{15}, \qquad f_6(x) = 0, \qquad f_7(x) = \frac{17}{315},$$

$$\begin{aligned} A_1 &= -120(f_1^2 f_2 + f_0 f_2^2 + 2f_0 f_1 f_3 + f_0^2 f_4)t^4 - 120(f_1 f_2^2 + f_1^2 f_3 + 2f_0 f_2 f_3 + 2f_0 f_1 f_4 + f_0^2 f_5)t^5 - 40(f_2^3 + 6f_1 f_2 f_3 + 3f_0 f_3^2 \\ &\quad + 3f_1^2 f_4 + 6f_0 f_2 f_4 + 6f_0 f_1 f_5 + 3f_0^2 f_6)t^6 - 120(f_2^2 f_3 + f_1 f_3^2 + 2f_1 f_2 f_4 + 2f_0 f_3 f_4 + f_1^2 f_5 + 2f_0 f_2 f_5 + 2f_0 f_1 f_6 + f_0^2 f_7)t^7 \end{aligned}$$

$$\begin{aligned} B_1 &= 120f_0(-f_1^4 - 6f_0 f_1^2 f_2 - 2f_0^2 f_2^2 - 4f_0^2 f_1 f_3 - f_0^3 f_4)t^4 + -24(f_1^5 + 20f_0 f_1^3 f_2 + 30f_0^2 f_1 f_2^2 + 30f_0^2 f_1^2 f_3 + 20f_0^3 f_2 f_3 \\ &\quad + 20f_0^3 f_1 f_4 + 5f_0^4 f_5)t^5 + 120(-f_1^4 f_2 - 6f_0 f_1^2 f_2^2 - 2f_0^2 f_2^3 - 4f_0 f_1^3 f_3 - 12f_0^2 f_1 f_2 f_3 - 2f_0^3 f_3^2 - 6f_0^2 f_1^2 f_4 - 4f_0^3 f_2 f_4 \\ &\quad - 4f_0^3 f_1 f_5 - f_0^4 f_6)t^6 + 120(-2f_1^3 f_2^2 - 4f_0 f_1 f_2^3 - f_1^4 f_3 - 12f_0 f_1^2 f_2 f_3 - 6f_0^2 f_2^2 f_3 - 6f_0^2 f_1 f_3^2 \\ &\quad - 4f_0 f_1^3 f_4 - 12f_0^2 f_1 f_2 f_4 - 4f_0^3 f_3 f_4 - 6f_0^2 f_1^2 f_5 - 4f_0^3 f_2 f_5 - 4f_0^3 f_1 f_6 - f_0^4 f_7) \end{aligned}$$

$$\begin{aligned} u_2(x,t) &= -L^{-1}(A_1 + B_1 + Ru_1) \\ &= \frac{62}{2835}t^9 + \frac{1382}{155925}t^{11} \\ &= f_8(x)t^8 + f_9(x)t^9 + f_{10}(x)t^{10} + f_{11}(x)t^{11}, \\ &\ \ \vdots \end{aligned}$$

Following the same procedures we obtain

$$\begin{aligned} u_3(x,t) &= -L^{-1}(A_2 + B_2 + Ru_2) \\ &= \frac{124}{6081075}t^{13} + \frac{4}{638512875}t^{15}, \\ &\ \ \vdots \end{aligned}$$

And so on….

Considering these components, the solution can be approximated as:

$$u(x,t) \simeq \phi_n(x,t) = \sum_{m=0}^{n} u_m(x,t). \tag{60}$$

$$\begin{aligned}
\phi_1 &= t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7,\\
\phi_2 &= t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{1382}{155925}t^{11},\\
\phi_3 &= t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{1382}{155925}t^{11} + \frac{124}{6081075}t^{13} + \frac{4}{638512875}t^{15},\\
&\ \vdots
\end{aligned} \tag{61}$$

ϕ_n contains the exact power series expansion of the closed form solution

$$u(t) = \tan(t). \tag{62}$$

Using ADM

$$\begin{aligned}
u_1(x,t) &= L^{-1}(A_0 + B_0 + Ru_0)\\
&= \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{4}{189}t^9 + \frac{2}{297}t^{11} + \frac{19}{11583}t^{13} + \frac{2}{7371}t^{15} + \frac{1}{38556}t^{17} + \frac{1}{941868}t^{19},\\
u_2(x,t) &= L^{-1}(A_1 + B_1 + Ru_1)\\
&= \frac{2}{2835}t^9 + \frac{332}{155925}t^{11} + \frac{113}{57915}t^{13} + \frac{1357}{1216215}t^{15} + \frac{254273}{578918340}t^{17}\\
&\quad + \frac{40217}{314269956}t^{19} + \frac{502919}{16499172690}t^{21} + \frac{14320463}{2276885831220}t^{23} + \cdots,\\
u_3(x,t) &= L^{-1}(A_2 + b_2 + Ru_2)\\
&= \frac{4}{6081075}t^{13} + \frac{43894}{638512875}t^{15} + \frac{90362}{723647925}t^{17} + \frac{2693239}{24748759035}t^{19}\\
&\quad + \frac{315446071}{5197239397350}t^{21} + \frac{129582991}{5009148828684}t^{23} + \cdots,
\end{aligned}$$

And so on....

Considering these components, the solution can be approximated as:

$$u(x,t) \simeq \phi_n(x,t) = \sum_{m=0}^{n} u_m(x,t). \tag{63}$$

$$\begin{aligned}
\phi_1 &= t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \underline{\frac{4}{189}t^9 + \frac{2}{297}t^{11} + \frac{19}{11583}t^{13} + \frac{2}{7371}t^{15} + \frac{1}{38556}t^{17} + \frac{1}{941868}t^{19} \ldots},\\
\phi_2 &= t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{1382}{155925}t^{11} + \underline{\frac{113}{57915}t^{13} + \frac{1357}{1216215}t^{15}}\\
&\quad \underline{+ \frac{254273}{578918340}t^{17} + \frac{40217}{314269956}t^{19} + \frac{502919}{16499172690}t^{21} + \frac{14320463}{2276885831220}t^{23} \ldots},\\
\phi_3 &= t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{1382}{155925}t^{11} + \frac{124}{6081075}t^{13} + \underline{\frac{4}{638512875}t^{15}}\\
&\quad \underline{+ \frac{90362}{723647925}t^{17} + \frac{2693239}{24748759035}t^{19} + \frac{315446071}{5197239397350}t^{21} + \frac{129582991}{5009148828684}t^{23} + \cdots},\\
&\ \vdots
\end{aligned}$$

As can be seen in ADM results, there are unsettled terms in ϕ_n.

3. Conclusion and summary

From analyzing the obtained results and the procedures used in the Adomian decomposition method (ADM) and the improved Adomian decomposition method (IADM). The following results are observed:

1. When $s = 1$ in Eq. (2), the ADM results are the same as the IADM results.
2. When $s = 2, 3, 4, \ldots$ in Eq. (2), the IADM results are more accurate than the ADM results and more convergent.
3. IADM eliminates the calculations of all the inaccurate terms in ADM.
4. The error of the truncated series solution obtained by ADM or IADM is very small near the initial point but the error increases as we are far from the initial point for more details see [18,25].

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